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In This Issue ...

Unit Fractions: Inception and Use

James Lowdermilk

The ancient Egyptians used the mathematical construct that we call unit fractions to perform arithmetical division. Unit fractions are fractions with a numerator of one, and they were added together by Egyptian scribes to solve division problems, such as

$$\frac{4}{5} | \frac{1}{2} 2 \frac{1}{5} 2 \frac{1}{10}.$$

These methods will be analyzed below. In modern mathematics we use common fractions, ratios or decimal representations to achieve the same means, such as

$$\frac{4}{5} | 4:5 | 0.8.$$

The Egyptians' chosen method to perform division has been called cumbersome and laborious because this method is difficult and hard for an unaccustomed mind to decipher. The scribes who used these methods understood their applications and were accomplished in their use. However, it has been questioned why the Egyptians would choose this difficult method over the simpler choice of common fractions and why unit fractions persisted throughout pharaonic times.

The cattle herders who roamed the humid prehistoric Sahara mingled and eventually merged with the agrarian settlers of the Nile valley. The traditions of the cattle herding nomads go back at least 12,000 YBP (Years Before Present), soon after the rains began to fall on the sands of the Sahara creating pasturelands. The domestication of cattle around this time (Midant-Reynes 2000:89) presented these people with the need to understand larger numbers in order to manage their herds. They also would have used the stars to navigate their herds to the proper pastures during the rainy or dry seasons. Seasonal lakes brought many of these nomadic tribes together every spring at the Nabta Playa depression in southern Egypt.

Standing stones have been discovered at Nabta Playa that date to the late 5th millennium BCE. These stones are aligned to the rising points of various stars during that epoch (Malville 1998:488). The alignments gave the people who erected these stones a referenced, unchanging horizon for viewing the stars. When a star is aligned with the stones an observation can be made. When this observation takes place as the sun rises, we call this a heliacal rising (figure 1).

Once a year, each star will rise above the observed stone alignment within minutes before the sun rises. The number of sunrises until this referenced heliacal rising is observed again can then be counted. The first thing noticed about this count is that each star's heliacal rising occurs 365 days apart on most occasions. Very soon it could be confirmed that almost every fourth rising takes an extra day, or 366 days – what we call a leap year. Quickly thereafter, it would be noticed that some stars occasionally only require three years to achieve a leap year while others might sometimes require five years. The high number of stone alignments at Nabta Playa reflects the inhabitants' attempts to collect and analyze this data.

As the observations continued year after year, every star would exhibit the 365, 365, 365, 366-day per observed year pattern. Slowly each star would break this pattern with an occasional early leap year, or late leap year, only to return to the pattern of a leap year every fourth year. By sheer coincidence the brightest star in the sky resided in just the right location for it to almost never break its four-year leap cycle. The location of the star we call Sirius caused the time from one heliacal rising to the next to be within seconds of 365 +1/4 days (Ingham 1969:36). Other stars had rising times minutes away from the quarter day behavior of Sirius, making this brightest star unique. In predynastic times Sirius was positioned so that it would return according to its expected four-year pattern for more than a thousand years.

A 1st Dynasty tablet bears the image of a recumbent cow, the goddess Isis-Sothis, representing this star. It reads, "Sothis [Sopdet], the opener of the year" (Parker 1950:34). The depictions of Sopdet as a cow thus identify the original worshipers of this star as the cattle herders who frequented the Nabta Playa depression and erected the stones aligned to this and many other stars. Another artifact connecting cattle to the stars is the Naqada III period "Hathor" palette (Midant-Reynes 2000:193-4). Five stars surround the head of a bull; one of these stars rests on the bull's head just as a star would appear as it rises above a stone alignment (figure 2). The people who erected the standing stones recognized the star Sirius/Sopdet for never breaking its four-year pattern and it was probably these cattle herders who began worshiping this star as Sopdet. The coincidence of the rising of Sirius/Sopdet

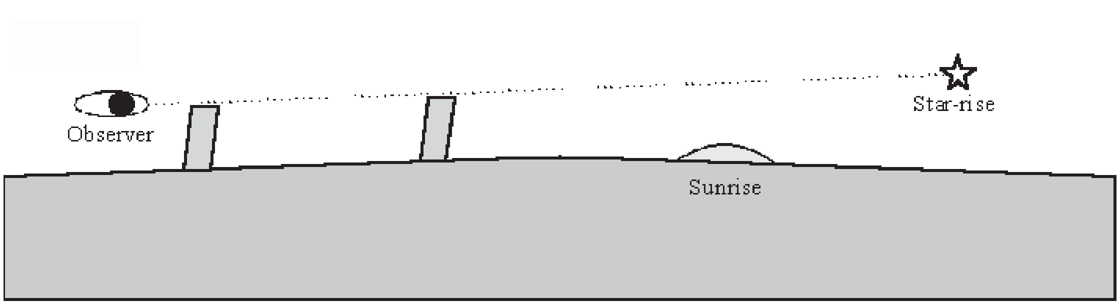

FIGURE 1: Using standing stones to determine when a star appears above the horizon. A star's heliacal rising occurs when it first appears each year.

with the Nile flood was recognized later when these cattle herders merged with the inhabitants of the Nile valley.

While few stars maintain a four-year leap-cycle, most routinely break their cycle at fairly regular intervals. A process we call precession of the equinox is the cause of this and is also responsible

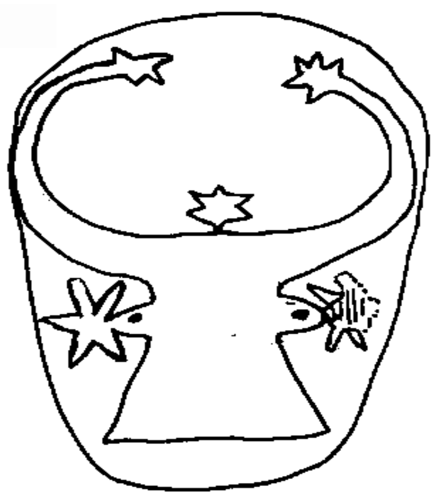

FIGURE 2: Naqada III period "Hathor" palette showing the connection between cattle and stars

for these stars' rising points to move over thousands of years to new rising points today. The ancient stone alignments are no longer aligned. The effects of precession caused Sirius to maintain its 365.25-day year in predynastic times. Near the celestial equator, the effects of precession are canceled out and each star in this region of the sky maintains a year equal to the Sidereal year of 365.2563 days. This value is within seconds of 365 + 10/39 days. This means that these stars would achieve 10 leap years every 39 years or that every 10th leap year comes on the third year of a cycle. The cattle herders making these observations noticed this and worked this information into a new calendar, which we now know as the Egyptian civil calendar (Lowdermilk 2000:9).

The Egyptian calendar counts 365 days every year and does not take leap years into account as our calendar does. When an observed event exhibits a leap year, its date on the Egyptian calendar changes by one day. Therefore, most events on the calendar move one calendar day every fourth year. The calendar was broken into weeks of 10 days each, called decans. The choice of 10-day decans reflects the recognition of the 10 leap years every 39-year cycle exhibited by the equatorial stars. When an observed star moves to the end of its decanal period its date changes after only 3 years to move into the next "week" (figure 3).

Sirius, or Sopdet, and the stars residing close to the celestial equator follow regular, recognizable patterns. Every other star in the sky has its own unique year, and in order to realize the simpler patterns, the entire sky must be analyzed. Sopdet was worshiped for its unique cycle of 365 + 1/4 days each year. This formed a baseline for the analysis of all the other stars. The proper mathematical tool for this investigation is unit fraction division.

Every year has 365 days and a small, additional part of a day. That additional fraction of a day adds up, year after year. When it adds up to more than one day, a leap year with one additional day occurs. This analysis is obvious in the simple case of Sopdet's extra 1/4-day each year, which gives one whole extra day in an interval of 4 years. The question is why, when other stars are observed, does this pattern sometimes break early, in a third year, or sometimes late, in a fifth year. The answer is that most star's years do not exactly equal 365 + 1/4 days, and the trick is to find out by how much each star's year is off that standard.

If a star breaks pattern by having a three year interval between leap years, then its own year is longer than 365 + 1/4 days. Conversely, if a star breaks pattern by having a five year interval between leap years, then its own year is shorter than 365 + 1/4 days. In the situation where the break comes regularly, as when every 39th year brings a three year interval, the difference is 1/4 times 1/39, making that star-year

$$365\ 2\ \frac{1}{4}\ 2\ \frac{1}{4\ 1\ 39}$$

days long. In the situation where the break does not come regularly, the number 39 above has to be replaced with the average number of years between breaks. The data needed to perform this analysis requires at least 100 to 150 years of accurate record keeping.

Some time after the stones were erected at Nabta Playa, deep wells were dug, enabling some members from the tribes to reside there year round and maintain observations during the wet and dry seasons, while the rest of the herdsmen were tending to the cattle in other pastures (Wasylikowa 1997:933). The means used to maintain the necessary records can only be speculated upon without specific evidence. However, to create the calendar, the residents of Nabta Playa must have collected the data on the stars. Some form of unit fraction analysis was performed before the calendar was created and implemented. The creators of the calendar understood the workings of the calendar before they inaugurated it.

The Egyptians began counting the years of their calendar on the day of the Sothic rising, the heliacal rising of Sopdet (Clagett 1995:29). Over the years the Sothic rising wandered

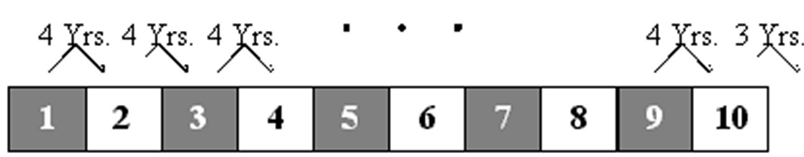

FIGURE 3: The Sidereal year starts one day later every 4th (or occasionally 3rd) Egyptian 365-day calender year.

off the first day of the calendar and then slowly returned, coinciding with the first day of their calendar again approximately 365*4 = 1,460 Egyptian years later, because almost every 4 years Sirius' rising would move one calendar day. This length of time is called a Sothic Cycle. According to Censorinus, writing in 239 CE, the first day of the Egyptian calendar coincided with the Sothic rising in 139 CE (Clagett 1995:307). Counting backwards by increments of 1,461 Julian years this coincidence of dates also occurred in 1321 BCE, 2782 BCE, and 4243 BCE. The 4243 BCE date corresponds to radiocarbon dating of sacrificial cattle burials at Nabta Playa (Malville 1998:488) and is probably very near the starting date of the calendar.

The count of years the calendar ran would have first been kept by the tribesmen who devised the calendar and then by the priests who took charge when Egypt was unified. The astronomer Harkhebi tells us in an inscription on his statue (c. 600 BCE) that he was "one who announces the rising of Sothis at the beginning of the year and then observes her on her first festival day, calculating her course at the designated times, observing what she does every day; everything she has ordered is in his charge" and he "does not disclose at all concerning his report." (Clagett 1995:495-6 v.2)

Evidence of knowledge of the workings of the calendar being held secret is also found in the Reisner papyrus, c.1900 BCE. If the Egyptian calendar year of 365 days is 10/39ths of a day short of a sidereal year, then it takes 39÷10 = 3.9 years for the calendar to lose one day to the sidereal year, not exactly 4 calendar years. In the Reisner papyrus, a hired scribe wrote the approximation 39÷10 = 4 even though elsewhere in the papyrus he has correctly worked the problems 30÷10 and 9÷10, which when added together give the correct value of 39÷10, proving his ability (Gillings 1972:221). Apparently the author of the Reisner papyrus knew or was told that the calculation 39÷10 was not to be performed in such a profane location as the official registers of a dockyard workshop.

Furthermore, when the geographer Strabo (2nd century CE), wrote of Plato's and Eudoxus' studies in Egypt in the 4th century BCE, he tells us that the Egyptian priests "did teach them the fractions of the day and the night which, running over and above the 365 days, fill out the time of the true year." (Strabo, Geography, p.83-5) These priests understood that the "true year" contains 10/39ths of a day more than the 365-day calendar year they used, but they were "secretive and slow to impart" this knowledge.

The cattle herders of the prehistoric Sahara would have been familiar with numbers on the order of a few hundred to keep count of the many tribes' herds. When collecting the data from the stars aligned with the standing stones they would first need numbers less than one thousand. In analyzing this data they would need to investigate larger numbers. By the time of Narmer (c. 3000 BCE) they had established numbers on the order of millions. The count of booty taken when Narmer conquered Lower Egypt is found on a mace head with his name. It shows that 1,422,000 goats, 400,000 oxen, and 120,000 prisoners were captured (Clagett 1989:6). These counts were undertaken not only to collect new data about their world, but also the people were taught to count so that those who enjoyed working with numbers and excelled could be identified and educated in the higher mathematics of fractions and calendars. The ruler could then utilize and exploit their talents for his benefit.

Following the creation of unit fraction division by one man or a small group working together, future generations had to be taught how to work with this difficult mathematical tool. The calendar was in use throughout pharaonic Egypt and its maintenance required the use of unit fractions. For this reason unit fractions were taught and used throughout pharaonic times. The Rhind papyrus (circa 1500 BCE) takes the form of a mathematical primer used to teach methods of unit fraction division, among other mathematical tools. The papyrus provides examples of mathematical problems without the benefit of a written explanation. The priests who specialized in this branch of teaching provided explanations orally.

The Rhind papyrus is a copy of a papyrus written 300 years earlier (Clagett 1999:113). An oral tradition would have accompanied this papyrus for every generation of those 300 years, continuing on with the new copy. This may indicate that the traditional method of teaching unit fractions was oral all the way back to the first use of unit fractions near the beginning of their calendar, prior to the implementation of hieroglyphic writing. The 300 years the previous copy of this papyrus was in use also reveals the working life of a papyrus used as a high school or college equivalent text.

The papyrus begins with the division problems 2÷3, 2÷5, 2÷7, ... 2÷101. These provide examples of how to perform division when the divisor grows larger and larger. The next examples in the papyrus show 1÷10, 2÷10, 3÷10, ... 9÷10. These problems show how to treat a quotient, as the dividend grows larger. The methods of unit fraction division have been called cumbersome and laborious. Once mastered, unit fraction division is no more cumbersome and laborious than modern long division.

Each division problem performed by the ancient Egyptians was always accompanied by ancillary numbers written below the problem, with one ancillary number for each unit fraction in the answer. For example:

$$\frac{11}{15} \mid \frac{1}{2}\,2\,\frac{1}{5}\,2\,\frac{1}{30}$$

$$\left(7\frac{1}{2}\right) \; \Psi\beta \; \left(\frac{1}{2}\right).$$

These numbers are built so that when each of them is multiplied with the denominator of its corresponding unit fraction, the denominator of the original problem is the result. When all the ancillary numbers are added, the result is the numerator of the original problem. These numbers are not unlike the subtractions undertaken in a modern long division problem. They are used to determine the next step in the problem. The ancillary numbers in the beginning steps of unit fraction division are built to get the result close to the correct answer. The ancillary numbers near the end are used to zero in on the correct result. This method is only one means to divide numbers into a unit fraction answer.

An interesting aspect of unit fraction division is that answers to division problems are not unique – for example 11/15 can also be expressed as:

$$\frac{11}{15} \mid \frac{1}{2}\,2\,\frac{1}{6}\,2\,\frac{1}{15}.$$

Beginning certain answers with

$$\frac{1}{2}\,2\,\frac{1}{6}$$

makes many problems easier because that value is equal to 2/3. The Egyptian mathematicians realized this and preferred the use of 2/3 in place of 1/2+1/6 for ease and brevity in their work.

A good first step to any division problem is to quickly estimate whether the answer is bigger than 2/3, 1/2, 1/3, 1/4, etc. The largest of these values is often but not always the preferred choice for the first fraction. Some fractions, such as 1/7th, are difficult in most problems so they can be avoided by using the next smaller fraction, such as 1/8th. These hints are only gleaned from continued practice, without the aid of a calculator, sometimes without pencil and paper. Some proficient Egyptian scribes would have excelled at unit fraction division.

Unit fraction division was taught in the schools and temples of ancient Egypt. In Mesopotamia, when division problems were beyond the reach of a scribe, he would consult a division table written on a clay tablet to find the answer. A base-60 number system works well with division tables because the number 60 is an abundant number, containing many divisors. The methods of unit fraction division inherently do not lend themselves to reference tables of division and must have been performed for each problem by a competent scribe. The Mathematical Leather Roll in the British Museum suggests tables of addition and subtraction of fractions existed, but they have not been found. Division tables clearly did not exist (Gillings 1972:11-12).

People with curiosity about and talent in unit fractions would have investigated their structure. The Wedjat eye is an example of the results of their investigations. The Wedjat eye drawn as a right eye represents the sun and drawn as a left eye represents the moon. The Wedjat eye is also an example of an infinite geometric series. Separate parts of the eye break down into the hieroglyphic signs for the numerical fractions 1/2, 1/4, 1/8, 1/16, 1/32, and 1/64. These numbers form a geometric sequence. When the obvious pattern is continued indefinitely, these numbers form a convergent infinite geometric series, or they all infinitely add up to a finite number, in this case one, i.e.

$$\frac{1}{2} 2 \frac{1}{4} 2 \frac{1}{8} 2 \cdots | 1 .$$

By representing the Wedjat series as a deeply religious symbol, the ancient Egyptians acknowledged their understanding of this mathematical situation. To deny the Egyptians ever gained this knowledge is to say they never investigated the nuances of the mathematical systems they used daily or that every Egyptian scribe was incapable of understanding the mathematics that they regarded so highly as to view it religiously. A text that confirms this understanding has never been found, but individual scribes with sufficient ability would have pondered on the significance of the Wedjat fractions. The Wedjat eye represents an infinite path that leads to one.

The mathematics of the series represented in the Wedjat eye implies possible investigations that involve infinity. Without further textual evidence we cannot know in what directions the Egyptians took their investigations of the valuable and interesting tool we call unit fraction division. Modern investigations of the underlying structures of unit fractions are found in upper level modern algebra texts under the guise of the multiplicative inverse of a number, the modern mathematical term for unit fractions. An example of the modern definition of multiplicative inverses is found in problem 9 of the Rhind papyrus. It states, using modern notation, that $4/7 * 7/4 = 1$. The definition of multiplicative inverses states that two numbers, a and b, are multiplicative inverses of each other if they multiply to equal one, or $a * b = 1$.

The adept and varied skills exhibited in the small number of surviving mathematical texts that contain examples of unit fraction division suggest that over the thousands of years the Egyptians used this tool they gained a deep understanding of their use and structure.

In conclusion, the cattle herders of the Sahara erected standing stones to collect data on the rising times of many stars. In trying to decipher this data, someone created a new tool called unit fraction division to correctly interpret the numbers. This understanding of the behavior of the stars led to the creation of the Egyptian calendar of 365 days each calendar year. The maintenance of this calendar required an understanding of unit fractions so the use of unit fractions persisted as long as the calendar was used. Unit fraction methods were taught to scribes throughout ancient Egyptian history. These mathematicians investigated the structures of unit fractions and became very accomplished in their use.

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